Lesson 10: Using Simulation to Estimate a Probability

Classwork

In previous lessons, you estimated probabilities of events by collecting data empirically or by establishing a theoretical probability model. There are real problems for which those methods may be difficult or not practical to use. Simulation is a procedure that will allow you to answer questions about real problems by running experiments that closely resemble the real situation.

It is often important to know the probabilities of real-life events that may not have known theoretical probabilities. Scientists, engineers, and mathematicians design simulations to answer questions that involve topics such as diseases, water flow, climate changes, or functions of an engine. Results from the simulations are used to estimate probabilities that help researchers understand problems and provide possible solutions to these problems.

Example 1: Families

How likely is it that a family with three children has all boys or all girls?

Let's assume that a child is equally likely to be a boy or a girl. Instead of observing the result of actual births, a toss of a fair coin could be used to simulate a birth. If the toss results in heads (H), then we could say a boy was born; if the toss results in tails (T), then we could say a girl was born. If the coin is fair (i.e., heads and tails are equally likely), then getting a boy or a girl is equally likely.

Exercises 1–2

Suppose that a family has three children. To simulate the genders of the three children, the coin or number cube or a card would need to be used three times, once for each child. For example, three tosses of the coin resulted in HHT, representing a family with two boys and one girl. Note that HTH and THH also represent two boys and one girl.

1. Suppose a prime number (P) result of a rolled number cube simulates a boy birth, and a non-prime (N) simulates a girl birth. Using such a number cube, list the outcomes that would simulate a boy birth, and those that simulate a girl birth. Are the boy and girl birth outcomes equally likely?

2. Suppose that one card is drawn from a regular deck of cards, a red card (R) simulates a boy birth and a black card (B) simulates a girl birth. Describe how a family of three children could be simulated.

Example 2

Simulation provides an estimate for the probability that a family of three children would have three boys or three girls by performing three tosses of a fair coin many times. Each sequence of three tosses is called a trial. If a trial results in either HHH or TTT, then the trial represents all boys or all girls, which is the event that we are interested in. These trials would be called a "success." If a trial results in any other order of H's and T's, then it is called a "failure."

The estimate for the probability that a family has either three boys or three girls based on the simulation is the number of successes divided by the number of trials. Suppose 100 trials are performed, and that in those 100 trials, 28 resulted in either HHH or TTT. Then the estimated probability that a family of three children has either three boys or three girls would be $\frac{28}{100} = 0.28$.

Exercises 3–5

3. Find an estimate of the probability that a family with three children will have exactly one girl using the following outcomes of 50 trials of tossing a fair coin three times per trial. Use H to represent a boy birth, and T to represent a girl birth.

HHT	HTH	HHH	TTH	THT	THT	HTT	HHH	TTH	HHH
HHT	TTT	HHT	TTH	HHH	HTH	THH	TTT	THT	THT
THT	HHH	THH	HTT	HTH	TTT	HTT	HHH	TTH	THT
THH	HHT	TTT	TTH	HTT	THH	HTT	HTH	TTT	HHH
HTH	HTH	THT	TTH	TTT	HHT	HHT	THT	TTT	HTT

4. Perform a simulation of 50 trials by rolling a fair number cube in order to find an estimate of the probability that a family with three children will have exactly one girl.
   a. Specify what outcomes of one roll of a fair number cube will represent a boy, and what outcomes will represent a girl.

b. Simulate 50 trials, keeping in mind that one trial requires three rolls of the number cube. List the results of your 50 trials.

c. Calculate the estimated probability.

5. Calculate the theoretical probability that a family with three children will have exactly one girl.

   a. List the possible outcomes for a family with three children. For example, one possible outcome is BBB (all three children are boys).

   b. Assume that having a boy and having a girl are equally likely. Calculate the theoretical probability that a family with three children will have exactly one girl.

   c. Compare it to the estimated probabilities found in parts (a) and (b) above.

Example 3: Basketball Player

Suppose that, on average, a basketball player makes about three out of every four foul shots. In other words, she has a 75% chance of making each foul shot she takes. Since a coin toss produces equally likely outcomes, it could not be used in a simulation for this problem.

Instead, a number cube could be used by specifying that the numbers 1, 2, or 3 represent a hit, the number 4 represents a miss, and the numbers 5 and 6 would be ignored. Based on the following 50 trials of rolling a fair number cube, find an estimate of the probability that she makes five or six of the six foul shots she takes.

441323	342124	442123	422313	441243
124144	333434	243122	232323	224341
121411	321341	111422	114232	414411
344221	222442	343123	122111	322131
131224	213344	321241	311214	241131
143143	243224	323443	324243	214322
214411	423221	311423	142141	411312
343214	123131	242124	141132	343122
121142	321442	121423	443431	214433
331113	311313	211411	433434	323314

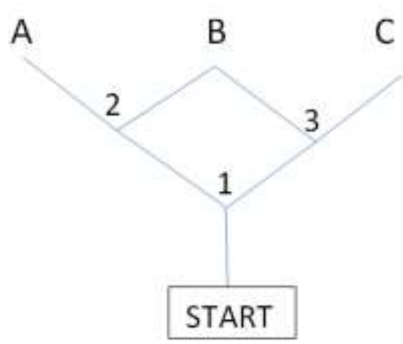
Lesson Summary

In previous lessons, you estimated probabilities by collecting data and found theoretical probabilities by creating a model. In this lesson you used simulation to estimate probabilities in real problems and in situations for which empirical or theoretical procedures are not easily calculated.

Simulation is a method that uses an artificial process (like tossing a coin or rolling a number cube) to represent the outcomes of a real process that provides information about the probability of events. In several cases, simulations are needed to both understand the process as well as provide estimated probabilities.

Problem Set

1. A mouse is placed at the start of the maze shown below. If it reaches station B, it is given a reward. At each point where the mouse has to decide which direction to go, assume that it is equally likely to go in either direction. At each decision point 1, 2, 3, it must decide whether to go left (L) or right (R). It cannot go backwards.



a. Create a theoretical model of probabilities for the mouse to arrive at terminal points A, B, and C.

   i. List the possible paths of a sample space for the paths the mouse can take. For example, if the mouse goes left at decision point 1, and then right at decision point 2, then the path would be denoted LR.

   ii. Are the paths in your sample space equally likely? Explain.

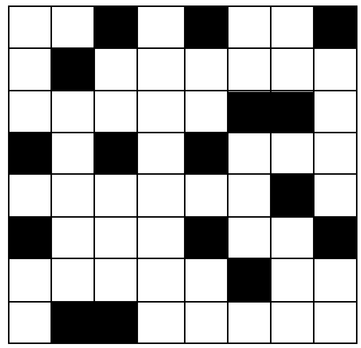
   iii. What are the theoretical probabilities that a mouse reaches terminal points A, B, and C? Explain.

b. Based on the following set of simulated paths, estimate the probabilities that the mouse arrives at points A, B, and C.

RR	RR	RL	LL	LR	RL	LR	LL	LR	RR
LR	RL	LR	RR	RL	LR	RR	LL	RL	RL
LL	LR	LR	LL	RR	RR	RL	LL	RR	LR
RR	LR	RR	LR	LR	LL	LR	RL	RL	LL

c. How do the simulated probabilities in part (b) compare to the theoretical probabilities of part (a)?

2. Suppose that a dartboard is made up of the 8×8 grid of squares shown below. Also, suppose that when a dart is thrown, it is equally likely to land on any one of the 64 squares. A point is won if the dart lands on one of the 16 black squares. Zero points are earned if the dart lands in a white square.



a. For one throw of a dart, what is the probability of winning a point? Note that a point is won if the dart lands on a black square.

b. Lin wants to use a number cube to simulate the result of one dart. She suggests that 1 on the number cube could represent a win. Getting $2, 3$, or 4 could represent no point scored. She says that she would ignore getting a 5 or 6. Is Lin's suggestion for a simulation appropriate? Explain why you would use it *or,* if not, how you would change it.

c. Suppose a game consists of throwing a dart three times. A trial consists of three rolls of the number cube. Based on Lin's suggestion in part (b) and the following simulated rolls, estimate the probability of scoring two points in three darts.

324	**332**	**411**	**322**	**124**
224	**221**	**241**	**111**	**223**
321	**332**	**112**	**433**	**412**
443	**322**	**424**	**412**	**433**
144	**322**	**421**	**414**	**111**
242	**244**	**222**	**331**	**224**
113	**223**	**333**	**414**	**212**
431	**233**	**314**	**212**	**241**
421	**222**	**222**	**112**	**113**
212	**413**	**341**	**442**	**324**

d. The theoretical probability model for winning 0, 1, 2, and 3 points in three throws of the dart as described in this problem is

i. winning 0 points has a probability of 0.42;

ii. winning 1 point has a probability of 0.42;

iii. winning 2 points has a probability of 0.14;

iv. winning 3 points has a probability of 0.02.

Use the simulated rolls in part (c) to build a model of winning $0, 1, 2$, and 3 points, and compare it to the theoretical model.